

A Lie algebra for Frölicher groups

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Abstract

Frölicher spaces form a cartesian closed category which contains the category of smooth manifolds as a full subcategory. Therefore, mapping groups such as $C^\infty(M, G)$ or $\text{Diff}(M)$, and also projective limits of Lie groups, are in a natural way objects of that category, and group operations are morphisms in the category. We call groups with this property Frölicher groups. One can define tangent spaces to Frölicher spaces, and in the present article we prove that, under a certain additional assumption, the tangent space at the identity of a Frölicher group can be equipped with a Lie bracket. We discuss an example which satisfies the additional assumption.

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0. Introduction

Given a category \mathcal{C} whose objects are sets together with some additional structure, a natural question is whether the morphism sets $\text{Hom}_{\mathcal{C}}(X, Y)$ can be equipped with that structure, thus making them objects of \mathcal{C} . For instance in the category of topological spaces, we can equip $C(X, Y)$ with the compact-open topology. If one restricts to compactly generated Hausdorff spaces and defines the topology on $X \times Y$ appropriately, there is a homeomorphism

$$C(X, C(Y, Z)) \cong C(X \times Y, Z), \quad (0.1)$$

which simply maps $\varphi : X \rightarrow C(Y, Z)$ to the map $\tilde{\varphi}(x, y) := \varphi(x)(y)$. Eq. (0.1) is called the *exponential law*, and categories in which a form of the exponential law holds are called *cartesian closed*.

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There have been efforts to define cartesian closed categories of vector spaces and differentiable mappings, leading to the convenient calculus of Kriegl, Michor and others. We refer the reader to the monograph [15] and the extensive list of references therein.

On the other hand, there are more abstract approaches to ‘sets with smooth structure’. For instance, there are the closely related *diffeological spaces* of Souriau and *differential spaces* of Chen. In the present paper we use a notion of smooth spaces due to Frölicher. The smooth structure of a *Frölicher space* is given by curves $c : \mathbb{R} \rightarrow X$ and functions $f : X \rightarrow \mathbb{R}$. The sets of curves and functions are required to determine one another by asking that f is a function if and only if $f \circ c$ is in $C^\infty(\mathbb{R}, \mathbb{R})$ for all curves, and vice versa, c is a curve if and only if $f \circ c$ is in $C^\infty(\mathbb{R}, \mathbb{R})$ for all functions f . Frölicher, diffeological and other smooth spaces are compared in a recent preprint by Stacey [20], and also independently in the author’s Ph.D. thesis.

Both diffeological and Frölicher spaces form cartesian closed categories which contain the category of smooth manifolds as a full subcategory. This fact has led to recent interest in smooth spaces by mathematicians working in the field of higher category theory. We refer the reader to the paper by Baez and Hoffnung [1] and references therein.

Another application is in the field of infinite-dimensional Lie theory, since mapping groups such as $\text{Diff}(M)$ and $C^\infty(M, G)$, for smooth manifolds M and Lie groups G , are in a natural way groups in these categories. The motivating question for the present paper is: How much of the classical Lie theory can be generalized to the category of Frölicher spaces?

Tangent spaces to diffeological spaces have been defined by Hector [12], and tangent spaces to Frölicher spaces have been defined by Frölicher [7]. The Frölicher tangent spaces are conceptually simpler, since tangent vectors are represented by equivalence classes of smooth curves, like in the classical case. Our goal in the present paper is to define the structure of a Lie algebra on the tangent space $T_e G$ at the identity of a Frölicher group. This will be possible only if $T_e G$ satisfies a certain additional condition, namely that it is in a natural way in bijection with its own tangent space at 0. We verify this condition in a non-trivial example.

Let us mention that Frölicher groups have been investigated by Batubenge and Ntumba [2]. They show that for Frölicher groups G , the tangent space $T_e G$ can be identified with the space of left invariant vector fields. In the present article, we identify $T_e G$ with derivations of the algebra of smooth functions on G , and the main task is to define an element $[v, w] \in T_e G$ which corresponds to the commutator of the derivations associated with v and w . Diffeological groups have been investigated by Souriau [19], Donato and Iglesias [6], Hector and Macías-Virgós [13] and Leslie [16].

We briefly outline the content of the following sections.

In the first section, we define Frölicher spaces and then discuss differentiation theory in topological vector spaces. Topological vector spaces yield a family of examples for Frölicher spaces, and the particular example of $C^\infty(\mathbb{R}, \mathbb{R})$ plays a role in the definition of a Frölicher structure on Hom-sets. We state the most important properties of the category of Frölicher spaces: the existence of limits and colimits, and cartesian closedness. Finally, we give examples of Frölicher groups which are *not* Lie groups, thereby motivating the notion of Frölicher groups.

The second section is concerned with the tangent functor for Frölicher spaces. We define tangent spaces at a point, and define a Frölicher structure on the disjoint union of all tangent spaces. If X and Y are Frölicher spaces, we show that $T(X \times Y)$ is isomorphic to $TX \times TY$, and we show that TX agrees with the classical tangent bundle if X is a smooth finite-dimensional manifold.

The third section is the central part of the present article. If G is a Frölicher group, we write TG and T^2G as products, using the canonical group structure of TG . This is entirely analogous

to the classical case. If $\mathfrak{g} := T_e G$, we associate with any two vectors $v, w \in \mathfrak{g}$ a certain element $\xi \in T_0 \mathfrak{g}$. If a canonical map $\mathfrak{g} \rightarrow T_0 \mathfrak{g}$ is bijective, we let $[v, w]$ denote the inverse image of ξ under this map. Then we show that $(v, w) \mapsto [v, w]$ is a Lie bracket. To this end, we associate with each $v \in \mathfrak{g}$ a derivation D_v of the algebra of smooth functions on G . The main task consists in proving that $D_{[v, w]} = [D_v, D_w]$, where on the right hand side we have the usual commutator of derivations. From this, and linearity and injectivity of $v \mapsto D_v$, it follows immediately that $[\cdot, \cdot]$ satisfies the axioms of a Lie algebra.

In Section 4 we discuss an example. The additive group \mathbb{R}^J with its product Frölicher structure is a Frölicher group. Under a certain restriction on the cardinality of J we can prove that $T_0 \mathbb{R}^J$ is isomorphic to \mathbb{R}^J as a Frölicher space. We use a result of Kriegl and Michor, based on a theorem of Mazur, which shows that functions $f : \mathbb{R}^J \rightarrow \mathbb{R}$ of the Frölicher structure factor through a countable product \mathbb{R}^{J_0} . Since \mathbb{R}^{J_0} is a Fréchet space, we can use results from the first section in order to conclude the proof.

In an appendix, we sketch the argument of Kriegl and Michor used in Section 4.

1. Frölicher spaces

First let us recall the definition and basic properties of the category of smooth spaces as introduced by Frölicher [7]. If A and B are sets, let $\text{Map}(A, B)$ denote the collection of *set maps* from A to B .

Definition 1.1. (a) A *Frölicher structure* on a set X is a pair (C, F) , where $C \subset \text{Map}(\mathbb{R}, X)$ and $F \subset \text{Map}(X, \mathbb{R})$, such that the following two conditions are satisfied:

- $C = \{c : \mathbb{R} \rightarrow X \mid (\forall f \in F) f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})\},$
- $F = \{f : X \rightarrow \mathbb{R} \mid (\forall c \in C) f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})\}.$

The elements of C and F are called *curves* and *functions* of the structure, or *smooth curves* and *functions*, respectively. The triple (X, C, F) is called a *Frölicher space*. We will usually omit C and F from the notation, if there is no danger of confusion.

(b) A *morphism* or *smooth map* between Frölicher spaces (X, C, F) and (Y, C', F') is a set map $\varphi : X \rightarrow Y$ such that $\varphi \circ c \in C'$ for all $c \in C$, or equivalently $f \circ \varphi \in F$ for all $f \in F'$.

(c) Frölicher spaces and smooth maps form a category, which we denote as \mathcal{F} .

Remark 1.2. Let X be a set. Given any set $A \subset \text{Map}(X, \mathbb{R})$, we let

$$S(A) := \{c \in \text{Map}(\mathbb{R}, X) \mid (\forall f \in A) f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})\}$$

be the *saturation* of A . Similarly, one can define a saturation $S(B) \subset \text{Map}(X, \mathbb{R})$ for $B \subset \text{Map}(\mathbb{R}, X)$. The condition for (C, F) to be a Frölicher structure then reads

$$(C, F) = (S(F), S(C)).$$

The category \mathcal{F} is closed under various category theoretical constructions, for instance inductive and projective limits, quotients and subsets. Before we state some properties of the category \mathcal{F} , we study Frölicher structures on topological vector spaces.

Differentiation in topological vector spaces. We will use a notion of maps of class C^k which is studied in detail, and in a more general context, by Bertram et al. [3].

Definition 1.3. Let E and F be topological vector spaces and $U \subset E$ open. Let $U^{[1]} := \{(x, v, t) \mid x \in U, x + tv \in U\} \subset U \times E \times \mathbb{R}$. Then we say that a map $f : U \rightarrow F$ is of class C^1 on U if there is a continuous map $f^{[1]} : U^{[1]} \rightarrow F$ such that

$$f(x + tv) - f(x) = tf^{[1]}(x, t, v)$$

for all $(x, t, v) \in U^{[1]}$. In other words, $f^{[1]}$ is a continuous extension of the difference quotient $t^{-1}(f(x + tv) - f(x))$. We say that f is of class C^2 if $f^{[1]}$ is of class C^1 , and inductively, f is of class C^{k+1} if it is of class C^k , and $f^{[k]}$ is of class C^1 . Finally, f is of class C^∞ if it is of class C^k for all $k \in \mathbb{N}$. We will frequently say ‘ f is C^k ’, and ‘ f is C^∞ ’, rather than f is of class C^k and of class C^∞ , respectively. We will not use the term *smooth* for C^∞ -maps, since it is already used for morphisms of Frölicher spaces.

Remark 1.4. The authors of [3] show that if the range is assumed to be a *locally convex* vector space, their definition of a C^k -map agrees with the more classical notion of differentiability in the sense of Michal and Bastiani (see [3, Proposition 7.4]). Note also that if the domain and range are Banach spaces, there is the notion of Fréchet differentiability, which however leads to the same notion of C^∞ -maps as Michal and Bastiani’s definition. Since we will only be interested in C^∞ -maps, the definition that we use generalizes the more classical notions.

The following theorem is classical. We will combine it with a more recent result to obtain information on Frölicher structures on metrizable topological vector spaces.

Theorem 1.5 (Boman’s Theorem). Let $f \in \text{Map}(\mathbb{R}^n, \mathbb{R})$ be such that $f \circ c$ is C^∞ whenever $c : \mathbb{R} \rightarrow \mathbb{R}^n$ is C^∞ . Then f is C^∞ .

Proof. See [4]. \square

As an immediate consequence of Boman’s theorem we have the following:

Corollary 1.6. If M is a smooth finite-dimensional manifold, then

$$(C^\infty(\mathbb{R}, M), C^\infty(M, \mathbb{R}))$$

is a Frölicher structure on M . The category of smooth finite-dimensional manifolds is a full subcategory of \mathcal{F} .

Theorem 1.7. Let E and F be topological \mathbb{R} -vector spaces. Let $U \subset E$ be open and $f : U \rightarrow F$ a mapping. If E is metrizable, then f is of class C^k if and only if the composition $f \circ c : \mathbb{R}^{k+1} \rightarrow F$ is of class C^k for every C^∞ -map $c : \mathbb{R}^{k+1} \rightarrow U$.

Proof. See [3, Theorem 12.4]. \square

Corollary 1.8. Let E be a metrizable topological vector space. Then $f : E \rightarrow \mathbb{R}$ is smooth if and only if for all smooth $c : \mathbb{R} \rightarrow E$, the composition $f \circ c$ is smooth.

Proof. Clearly, $f \circ c$ is smooth if both f and c are. Now suppose that f is smooth along smooth curves. Let $d : \mathbb{R}^{k+1} \rightarrow E$ be smooth. Then by assumption, the composition $f \circ d \circ c$ is smooth for all smooth c . Boman's Theorem implies that the map $f \circ d$ is smooth. By Theorem 1.7, it follows that f is of class C^k . Since k was arbitrary, f is smooth. \square

Remark 1.9. The above corollary says that for metrizable topological vector spaces E , the saturation $\mathcal{S}(C^\infty(\mathbb{R}, E))$ equals $C^\infty(E, \mathbb{R})$. However, $\mathcal{S}(C^\infty(E, \mathbb{R}))$ does not equal $C^\infty(\mathbb{R}, E)$ in general. Indeed, consider the vector space $E = L^{\frac{1}{2}}[0, 1]$ with the metric $d(f, g) = \int_0^1 |f(x) - g(x)|^{1/2} dx$. One can show that the continuous dual E^* consists only of the zero functional. Therefore, all smooth functions $f : E \rightarrow \mathbb{R}$ are constant, and $\mathcal{S}(C^\infty(E, \mathbb{R}))$ consists of all functions $c : \mathbb{R} \rightarrow E$.

There is an important class of topological vector spaces E on which there is a Frölicher structure of the form $(C^\infty(\mathbb{R}, E), F)$.

Definition 1.10. A locally convex vector space E is *convenient* if the following holds: Let $c \in \text{Map}(\mathbb{R}, E)$ be a map such that $\varphi \circ c$ is C^∞ for all continuous linear functionals φ on E . Then c is of class C^∞ .

This is only one of several equivalent characterizations of convenient locally convex spaces; see [15, Theorem I.2.14] for more. The following theorem is due to Alfred Frölicher.

Theorem 1.11. *Fréchet spaces are convenient.*

Proof. See [8, Théorème 1]. \square

Corollary 1.12. *If E is a Fréchet space, then the pair $(C^\infty(\mathbb{R}, E), C^\infty(E, \mathbb{R}))$ is a Frölicher structure on E .*

Proof. Fréchet spaces are metrizable and convenient, so this follows from Corollary 1.8 and Theorem 1.11. \square

Example 1.13. The vector space $E = C^\infty(\mathbb{R}, \mathbb{R})$ is a Fréchet space when equipped with the topology of compact convergence of all derivatives. By Corollary 1.12, the pair $(C^\infty(\mathbb{R}, E), C^\infty(E, \mathbb{R}))$ is a Frölicher structure on E .

Properties of the category \mathcal{F} . We summarize some properties of the category \mathcal{F} . First we need to introduce the important concepts of initial and final Frölicher structures, which are analogous to initial and final topologies.

Definition 1.14. Given Frölicher structures $(C, F), (C', F')$ on a set X , we say that (C, F) is *finer* than (C', F') if $C \subset C'$ or equivalently if $F' \subset F$. Similarly, (C, F) is *coarser* than (C', F') if $C' \subset C$.

Lemma 1.15. *If X is a set and $\tilde{F} \subset \text{Map}(X, \mathbb{R})$, then there is a unique coarsest Frölicher structure (C, F) on X with $\tilde{F} \subset F$. It is given by $(\mathcal{S}(\tilde{F}), \mathcal{S}^2(\tilde{F}))$, where \mathcal{S} is the saturation from Remark 1.2. Similarly, given $\tilde{C} \subset \text{Map}(\mathbb{R}, X)$, the pair $(\mathcal{S}^2(\tilde{C}), \mathcal{S}(\tilde{C}))$ is the finest Frölicher structure on X for which elements of \tilde{C} are curves. We say that (C, F) is generated by \tilde{C} or \tilde{F} , respectively.*

Proof. It suffices to show that the given pairs are Frölicher structures, since they are then clearly the coarsest and finest containing \tilde{F} and \tilde{C} , respectively.

Let us only prove that $(\mathcal{S}(\tilde{F}), \mathcal{S}^2(\tilde{F}))$ is a Frölicher structure, the proof for the other pair being analogous. By Remark 1.2, we need to show that

$$(\mathcal{S}(\tilde{F}), \mathcal{S}^2(\tilde{F})) = (\mathcal{S}^3(\tilde{F}), \mathcal{S}^2(\tilde{F})).$$

In general we have $\mathcal{S}(A) \subset \mathcal{S}^3(A)$, and we want to prove equality. But since we also have $A \subset \mathcal{S}^2(A)$, this follows easily:

$$f \in \mathcal{S}^3(A) \Rightarrow \forall a \in A : f \circ a \in C^\infty(\mathbb{R}, \mathbb{R}) \Rightarrow f \in \mathcal{S}(A). \quad \square$$

The existence of finest and coarsest Frölicher structures containing a given set of maps allows us to make the following definitions.

Definition 1.16. (a) Let X be a set, $\{(X_i, C_i, F_i)\}_{i \in I}$ a collection of Frölicher spaces, and $g_i : X_i \rightarrow X$ and $f_i : X \rightarrow X_i$ set maps. The *initial Frölicher structure with respect to the maps f_i* is the Frölicher structure generated by all $f \circ f_i$ with $i \in I$ and $f \in F_i$. Similarly, the *final Frölicher structure with respect to the maps g_i* is the Frölicher structure generated by all $g_i \circ c$ with $i \in I$ and $c \in C_i$.

In particular, if X is a Frölicher space and $\iota : A \rightarrow X$ the inclusion of a subset and $\pi : X \rightarrow B$ the projection onto a quotient, then the *subset structure on A* is the initial structure with respect to ι , and the *quotient structure on B* is the final structure with respect to π .

(b) We refer the reader to [17] for the precise definition of diagram, limit and colimit in a category. Given a diagram with objects (X_i, C_i, F_i) in the category \mathcal{F} , we can form the limit X and colimit Y of the underlying sets. The limit and colimit come equipped with maps $a_i : X \rightarrow X_i$ and $\alpha_i : X_i \rightarrow Y$. If we equip the sets X and Y with the initial and final Frölicher structure with respect to the maps a_i and α_i , it is easy to show that one gets limit and colimit in the category of Frölicher spaces.

Example 1.17 (Product Structure). Consider the discrete diagram with objects $\{(X_i, C_i, F_i)\}_{i \in I}$. The limit of the underlying sets is the cartesian product $\prod_{i \in I} X_i$, and the *product Frölicher structure* is the coarsest for which all projections π_i are morphisms. By Lemma 1.15, its curves are precisely the maps $c = (c_i)_{i \in I} : \mathbb{R} \rightarrow \prod_{i \in I} X_i$ such that $c_i \in C_i$ for all $i \in I$.

Definition 1.18. Let $C^\infty(\mathbb{R}, \mathbb{R})$ carry the Frölicher structure as in Example 1.13. Given Frölicher spaces (X, C, F) and (Y, C', F') , equip the set $\text{Hom}(X, Y)$ with the initial Frölicher structure with respect to the collection $\{\Phi_{f,c} \mid f \in F', c \in C\}$ of maps

$$\Phi_{f,c} : \text{Hom}(X, Y) \rightarrow C^\infty(\mathbb{R}, \mathbb{R}), \quad \varphi \mapsto f \circ \varphi \circ c.$$

The following theorem says that the category \mathcal{F} is cartesian closed, and that composition yields smooth maps between Hom-objects.

Theorem 1.19. (a) If $\varphi \in \text{Hom}(X \times Y, Z)$, we let $\bar{\varphi} \in \text{Map}(X, \text{Map}(Y, Z))$ be defined by $\bar{\varphi}(x)(y) := \varphi(x, y)$. If φ is smooth, then so are all $\bar{\varphi}(x)$. Hence $\bar{\varphi}$ takes values in $\text{Hom}(Y, Z)$, and it is also smooth. The map

$$\text{Hom}(X \times Y, Z) \rightarrow \text{Hom}(X, \text{Hom}(Y, Z)), \quad \varphi \mapsto \bar{\varphi}$$

is an isomorphism of Frölicher spaces.

(b) If $\varphi : Y \rightarrow Z$ is smooth, then so is the map

$$L_\varphi : \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Z), \quad f \mapsto \varphi \circ f$$

of left composition with φ .

Proof. Part (a) is proven in [15, Theorem 23.2], and part (b) is an easy consequence; see [15, Corollary 3.13]. \square

Combining this with the fact that manifolds form a full subcategory of \mathcal{F} , we get natural Frölicher structures on the sets $C^\infty(M, N)$ and $\text{Diff}(M)$ for manifolds M and N . A map φ from a Frölicher space X into one of those sets is smooth if and only if $\tilde{\varphi}(x, y) := \varphi(x)(y)$ is smooth on $X \times M$.

Examples of Frölicher groups. Let G be a group with multiplication m and inversion i , and let (C, F) be a Frölicher structure on G . We say that G is a *Frölicher group* if i is smooth, and if $m : G \times G \rightarrow G$ is smooth with respect to the product structure on $G \times G$.

We discuss some classes of examples.

Example 1.20 (Mapping Groups). Let X be a Frölicher space and G a Frölicher group with multiplication m_G and inversion i_G . Equip $\mathcal{G} := \text{Hom}(X, G)$ with its Frölicher structure from Definition 1.18, and the group structure given by pointwise multiplication and inversion. We have

$$fg = m_G \circ (f, g)$$

and

$$f^{-1} = i_G \circ f.$$

Thus, inversion and multiplication in \mathcal{G} are smooth by Theorem 1.19(b).

Example 1.21 (Diffeomorphism Groups). Let M be a Banach manifold. Equip $\text{Diff}(M) \subset \text{Hom}(M, M)$ with the subset Frölicher structure. The group multiplication is given by composition and therefore smooth by Theorem 1.19. Smoothness of inversion is implied by the following theorem, which can be found in the forthcoming book by Glöckner and Neeb [11]. For convenience, we will give the proof.

Theorem 1.22. *Let N be a locally convex manifold and M a Banach manifold. Let $f : N \rightarrow \text{Diff}(M)$ be a map such that $f_1 = \tilde{f} : N \times M \rightarrow M$ is C^∞ . Then the map $f_2 : (p, m) \mapsto f(p)^{-1}(m)$ is also C^∞ .*

Proof. Assume without loss of generality that N is open in some locally convex space E . Assume that M is modeled on the Banach space F . Fix $(p_0, m_0) \in N \times M$ and let $m'_0 = f(p_0)(m_0)$. We can find charts (φ, V) and (φ', V') around m_0 and m'_0 , and an open neighborhood U of p_0 such that $f_1(U \times V) \subset V'$. We define

$$\Phi := \varphi' \circ f_1 \circ (\text{id}_U \times \varphi^{-1}),$$

which is smooth from an open subset of $E \times F$ to F , and if we let $\Phi_{p_0}(x) := \Phi(p_0, x)$, then this is a local diffeomorphism, since $\Phi_{p_0} = \varphi' \circ f(p_0) \circ \varphi^{-1}$. In particular, the differential $d_x \Phi_{p_0}$ is invertible for $x \in \varphi(V)$. Now Glöckner's generalized implicit function theorem [10, Theorem 2.3] implies the existence of open neighborhoods U_1 of p_0 and V_1 of m_0 such that:

- $W = \cup_{p \in U_1} \{p\} \times \Phi_p(\varphi(V_1))$ is open in $E \times F$.
- The map $(p, m) \mapsto (p, \Phi_p(m))$ is a diffeomorphism of $U_1 \times \varphi(V_1)$ onto W .
- The map $(p, v) \mapsto \Phi_p^{-1}(v)$ is smooth on W .

The last item implies that $f_2(p, m) = (\varphi^{-1} \circ \Phi_p^{-1} \circ \varphi')(p, m)$ is smooth in a neighborhood of (p_0, m'_0) . \square

Example 1.23 (*Limits of Lie Groups*). It follows from Corollary 1.6 that all finite-dimensional Lie groups are Frölicher groups. Given a diagram of Frölicher groups, we can form the limit in the category of groups and in the category of Frölicher spaces. The two limits have the same underlying set G , and we claim that this limit is a Frölicher group. But the Frölicher structure is generated by the projections $\pi_i : G \rightarrow G_i$, which are also group homomorphisms. This readily implies that the group operations of G are smooth.

In particular, any product $\prod_{i \in I} G_i$ of Lie groups is a Frölicher group. We will consider the special case $(\mathbb{R}^J, +)$ in Section 4.

2. The tangent functor

The definitions of tangent and cotangent spaces for a Frölicher space go back to Frölicher [9]. If X is a Frölicher space, we equip the disjoint union TX of all tangent spaces with a natural Frölicher structure. We prove functoriality of $X \mapsto TX$, show that $T(X \times Y)$ is isomorphic to $TX \times TY$ and finally show that TM is diffeomorphic to the ‘classical’ tangent bundle for smooth finite-dimensional manifolds M .

Definition 2.1. (a) Let (X, C, F) be a Frölicher space and $x \in X$. We denote by C_x the set of curves satisfying $c(0) = x$. We introduce equivalence relations on C_x and F as follows:

- $c \sim d$ iff $(f \circ c)'(0) = (f \circ d)'(0)$ for all $f \in F$,
- $f \sim_x g$ iff $(f \circ c)'(0) = (g \circ c)'(0)$ for all $c \in C_x$.

(b) We define the *tangent space* and the *cotangent space* for X at x as $T_x X := C_x / \sim$ and $T^x X := F / \sim_x$, respectively. Note that $T^x X$ is a quotient vector space of F . We denote the equivalence classes of curves and functions by $[c]$ and $[f]$ respectively. We will frequently use the notation $v = [s \mapsto c(s)]$ for tangent vectors.

(c) Let $b : T_x X \times T^x X \rightarrow \mathbb{R}$ be given by $b([c], [f]) = (f \circ c)'(0)$. This is well-defined by the definitions of tangent and cotangent spaces. Clearly, b is linear in the second argument. By definition of the equivalence relations \sim and \sim_x , the map b induces an injection of $T_x X$ into $T^x X$.

(d) We say that $u = v + sw$ for $v, w \in T_x X$ and $s \in \mathbb{R}$ if $b(u, \xi) = b(v, \xi) + sb(w, \xi)$ holds for all $\xi \in T^x X$. The chain rule implies that if $v = [c]$, then the curve $t \mapsto c(st)$ represents sv . Thus, we can always multiply elements of the tangent space by scalars. We say that $T_x X$ is a *vector space* if $v + w$ exists for all tangent vectors $v, w \in T_x X$.

The following example illustrates that $T_x X$ is not always a vector space.

Example 2.2. If $X \subset \mathbb{R}^2$ is the coordinate cross with its subset Frölicher structure, consider tangent vectors $v, w \in T_{(0,0)} X$ represented by $t \mapsto (t, 0)$ and $t \mapsto (0, t)$ respectively. Then there is no vector u with $u = v + w$, since a curve $c : \mathbb{R} \rightarrow \mathbb{R}^2$ representing u would satisfy $c'(0) = (1, 1)$. But such a curve cannot map into X .

Lemma 2.3. *If G is a Frölicher group, then all tangent spaces $T_g G$ are vector spaces. In particular, the sum and additive inverse in $T_g G$ are given by*

$$\begin{aligned}[c] + [d] &= [s \mapsto c(s)g^{-1}d(s)] \\ -[c] &= [s \mapsto gc(s)^{-1}g].\end{aligned}\tag{2.1}$$

Proof. If $[c], [d] \in T_g G$, consider the smooth function $\gamma : (s, t) \mapsto c(s)g^{-1}d(t)$ from \mathbb{R}^2 into G . We get a curve through g after composing with the diagonal $\Delta : \mathbb{R} \rightarrow \mathbb{R}^2$. Let f be a function on G . The chain rule yields

$$(f \circ \gamma \circ \Delta)'(0) = \frac{\partial}{\partial s}(f \circ \gamma)(s, 0) + \frac{\partial}{\partial t}(f \circ \gamma)(0, t)$$

and, since $\gamma(s, 0) = c(s)$ and $\gamma(0, t) = d(t)$, the latter expression equals $(f \circ c)'(0) + (f \circ d)'(0)$, which proves that $\gamma \circ \Delta$ represents $[c] + [d]$. To check the formula for $-[c]$, just note that applying the sum formula one sees that $[c] + (-[c])$ is represented by

$$s \mapsto c(s)g^{-1}gc(s)^{-1}g = g.$$

This is the constant curve through g ; hence $[c] + (-[c])$ is the zero vector in $T_g G$. \square

Definition 2.4. The *tangent bundle* TX of a Frölicher space X is the disjoint union of all $T_x X$ for $x \in X$. If $\varphi : X \rightarrow Y$ is a smooth map between Frölicher spaces, we define $T\varphi : TX \rightarrow TY$ by $T\varphi([c]) = [\varphi \circ c]$.

Lemma 2.5. *Let $\varphi : X \rightarrow Y$ be a smooth map of Frölicher spaces. If $T_x X$ is a vector space, then vectors in the image of $T_x \varphi$ can be added, and $T_x \varphi$ is a linear map onto its image.*

Proof. Suppose that $[h] = [c] + [d] \in T_x X$. Then $T\varphi([h]) = [\varphi \circ h]$. Let f be smooth on Y . Then $f \circ \varphi$ is smooth on X ; hence $(f \circ \varphi \circ h)'(0) = (f \circ \varphi \circ c)'(0) + (f \circ \varphi \circ d)'(0)$ by assumption. This proves additivity, and it is also straightforward to see that $T_x \varphi$ commutes with scalar multiplication. \square

It is clear that for manifolds, the Frölicher tangent bundle agrees as a set with the classical tangent bundle. In particular, $T\mathbb{R}$ is identified with \mathbb{R}^2 via $[c] \mapsto (c(0), c'(0))$.

Given a Frölicher space (X, C, F) , we will from now on equip TX with the initial Frölicher structure with respect to all $Tf : TX \rightarrow T\mathbb{R}$ with $f \in F$.

Remark 2.6. One could also use the final structure on TX with respect to the differentials Tc of curves $c \in C$. Since all $Tf \circ Tc = T(f \circ c)$ are smooth, this would yield a *finer* structure than the initial structure with respect to the Tf .

We now characterize smooth curves into TX .

Definition 2.7. Let $c : \mathbb{R} \rightarrow TX$ be a smooth curve. For each $s \in \mathbb{R}$, pick a representative $c_s : \mathbb{R} \rightarrow X$ of $c(s)$. Then we say that

$$\begin{aligned}\gamma : \mathbb{R}^2 &\rightarrow X \\ \gamma(s, t) &:= c_s(t)\end{aligned}$$

represents c . The smooth curve $c : \mathbb{R} \rightarrow TX$ yields an element $\xi = [c] \in T^2 X$. We also say that ξ is represented by γ .

Remark 2.8. Suppose that $\xi \in T^2X$ is represented by $\gamma : \mathbb{R}^2 \rightarrow X$, and that $\varphi : X \rightarrow Y$ is smooth. Then

$$T^2\varphi(\xi) = [s \mapsto T\varphi([c_s])] = [s \mapsto [t \mapsto \varphi(\gamma(s, t))]]$$

which shows that $T^2\varphi(\xi)$ is simply represented by $\varphi \circ \gamma$.

Lemma 2.9. A map $\gamma : \mathbb{R}^2 \rightarrow X$ represents a smooth curve $c : \mathbb{R} \rightarrow TX$ if and only if it satisfies the following three conditions:

- (i) For fixed s , the map $t \mapsto \gamma(s, t)$ is smooth.
- (ii) The map $s \mapsto \gamma(s, 0)$ is smooth.
- (iii) The derivative $\frac{\partial(f \circ \gamma)}{\partial t}(s, 0)$ is a smooth function in s for all smooth functions f on X .

Proof. Clearly (i) is necessary in order to define $c : \mathbb{R} \rightarrow TX$ by $c(s) = [t \mapsto \gamma(s, t)]$. Now let $f : X \rightarrow \mathbb{R}$, and consider $Tf \circ c$. We have

$$Tf(c(s)) = \left(f(\gamma(s, 0)), \frac{\partial(f \circ \gamma)}{\partial t}(s, 0) \right)$$

after identifying $T\mathbb{R} \cong \mathbb{R}^2$. This shows that c is smooth if and only if (ii) and (iii) are satisfied. \square

Lemma 2.10. Let X and Y be Frölicher spaces, and $\varphi : X \rightarrow Y$ a smooth map. Then the projection $\pi : TX \rightarrow X$ and the differential $T\varphi : TX \rightarrow TY$ are smooth. In particular, $X \mapsto TX$ defines a functor $T : \mathcal{F} \rightarrow \mathcal{F}$.

Proof. Let $\pi : TX \rightarrow X$ and $\pi' : T\mathbb{R} \cong \mathbb{R}^2 \rightarrow \mathbb{R}$ be the projections of the tangent bundles. If f is a function on X , then $f \circ \pi = \pi' \circ Tf$. Hence $f \circ \pi$ is smooth for all such f ; therefore π is smooth. Now if $\varphi : X \rightarrow Y$ is smooth and f is a function on Y , then $T\varphi \circ Tf = T(\varphi \circ f)$ implies smoothness of $T\varphi$. \square

Proposition 2.11. For Frölicher spaces X and Y , the spaces $T(X \times Y)$ and $TX \times TY$ are isomorphic.

Proof. Let π_1, π_2 be the projections of $X \times Y$ onto X and Y , respectively, and write $c = (c_1, c_2)$ for curves into $X \times Y$. The map $\Pi := (T\pi_1, T\pi_2)$ is smooth by the previous lemma. It is onto, because $([c], [d]) \in TX \times TY$ is the image of $[s \mapsto (c(s), d(s))] \in T(X \times Y)$. Now let us show that Π is injective. Suppose that $\Pi([c]) = \Pi([d])$. Then $[c_i] = [d_i]$ for $i = 1, 2$. Let f be a smooth function on $X \times Y$. Write $c = c_1 \times c_2 \circ \Delta$, where $\Delta : \mathbb{R} \rightarrow \mathbb{R}^2$ is the diagonal map. Then

$$(f \circ c)'(0) = (f \circ (c_1 \times c_2) \circ \Delta)'(0) = \frac{\partial(f \circ (c_1, c_2))}{\partial s}(0, 0) + \frac{\partial(f \circ (c_1, c_2))}{\partial t}(0, 0)$$

by the chain rule. The derivatives on the right hand side can be rewritten as $(f_1 \circ c_2)'(0)$ and $(f_2 \circ c_1)'(0)$, where $f_1(y) := f(c_1(0), y)$ and $f_2(x) := f(x, c_2(0))$. The maps f_1 and f_2 are smooth on Y and X , respectively. Therefore, $(f_1 \circ c_2)'(0) = (f_1 \circ d_2)'(0)$ and similarly for f_2 . We conclude that

$$(f \circ c)'(0) = (f_1 \circ d_2)'(0) + (f_2 \circ d_1)'(0) = (f \circ d)'(0),$$

and hence $[c] = [d]$. It remains to prove that Π^{-1} is smooth. Let $c = (c_1, c_2) : \mathbb{R} \rightarrow TX \times TY$ be a curve. If c_i is represented by γ_i , then clearly $\Pi^{-1} \circ c$ is represented by

$(s, t) \mapsto (\gamma_1(s, t), \gamma_2(s, t))$. We use Lemma 2.9. Conditions (i) and (ii) are immediately verified. Property (iii) can be proven by a similar trick to the one used in the proof of the injectivity of Π , using the chain rule and the diagonal map, as well as the fact that property (iii) holds for the γ_i . \square

2.1. The tangent functor for finite-dimensional manifolds

The Frölicher tangent functor extends the classical tangent functor. In order to prove this, let us temporarily denote the Frölicher tangent functor by \tilde{T} and the classical one by T . Furthermore, fix a smooth n -dimensional manifold M with charts (U_i, φ_i) indexed by a set I . Then $\tilde{T}M$ can be described as set of equivalence classes $[x, i, a]$ where $x \in M$, $i \in I$ and $a \in \mathbb{R}^n$. Triples (x, i, a) and (y, j, b) are equivalent if $x = y$ and $d_{\varphi_i(x)}(\varphi_j \circ \varphi_i^{-1})(a) = b$. The manifold structure of TM is given by the charts $T\varphi_i : TU_i \rightarrow U_i \times \mathbb{R}^n$, $[x, i, a] \mapsto (\varphi_i(x), a)$.

We define a map $\Phi : \tilde{T}M \rightarrow TM$ as follows. If $[c] \in \tilde{T}M$, let $x = c(0)$ and choose $i \in I$ such that $x \in U_i$. Now define $a := (\varphi_i \circ c)'(0)$ and set $\Phi([c]) := [x, i, a]$.

Theorem 2.12. *The map $\Phi : \tilde{T}M \rightarrow TM$ is well-defined and a diffeomorphism of Frölicher spaces.*

Proof. Suppose that $[c] = [d]$. Then $c(0) = d(0)$, and we can pick $i \in I$ such that $\Phi([c]) = [x, i, a]$ and $\Phi([d]) = [x, i, b]$. It remains to show that $a = b$. We can use a smooth bump function ρ which is supported in U_i and is the constant function 1 on a smaller neighborhood of x , to define $f = \rho\varphi_i$. Since $[c] = [d]$, we get $a = (f \circ c)'(0) = (f \circ d)'(0) = b$. This proves well-definedness of Φ .

Next we check smoothness of Φ . Let $c : \mathbb{R} \rightarrow \tilde{T}M$ be a curve. Then $\pi \circ c : \mathbb{R} \rightarrow M$ is smooth, and we can choose a neighborhood U of $0 \in \mathbb{R}$ which is mapped into a chart U_i . Compose $\Phi \circ c : U \rightarrow TM$ with $T\varphi_i$ to get the map

$$s \mapsto ((\varphi_i \circ \pi \circ c)(s), (\varphi_i \circ c_s)'(0)),$$

where c_s represents the vector $c(s) \in \tilde{T}M$. This map is smooth by Lemma 2.9.

Let us construct an inverse map to Φ . If $U_i \times \mathbb{R}^n$ is a chart for TM , we map $(x, a) \in U_i \times \mathbb{R}^n$ to the vector in $\tilde{T}M$ represented by the curve $[t \mapsto \varphi_i^{-1}(x + ta)]$. This map is easily seen to be inverse to Φ . To see that it is smooth, recall that $\tilde{T}M$ carries the initial Frölicher structure with respect to the maps Tf for all smooth functions f on M . If $v \in T_x X$, we can represent v by a curve $t \mapsto c(t)$ with $c(0) = x$. By construction, $\Phi^{-1}(v) = [c]$, and $(Tf \circ \Phi^{-1})(v) = d_x f(v)$ is the classical differential. But df is smooth, which shows that Φ^{-1} is smooth. \square

3. The Lie algebra

Throughout this section, let G be a Frölicher group and $\mathfrak{g} = T_e G$. Let $v = [c]$ and $w = [d]$ be elements of \mathfrak{g} . Our strategy is to first define an element $\xi \in T_0 \mathfrak{g}$ which is related to the commutator of v and w in $T^2 G$. Then we consider a natural map $\mathfrak{g} \rightarrow T_0 \mathfrak{g}$; if this map is bijective, we let $[v, w]$ be the inverse image of ξ under this map. Lastly, we will view elements of \mathfrak{g} as derivations on the algebra F of functions on G to prove that $(v, w) \mapsto [v, w]$ is a Lie bracket.

Trivialization of tangent bundles. If G is a Frölicher group with multiplication m and inversion i , then the differentials Tm and Ti are smooth maps, and the chain rule $T(f \circ g) = Tf \circ Tg$ readily implies that TG is a Frölicher group with multiplication Tm and inversion Ti . We

simply write vw for $Tm(v, w)$ and note that $[c][d]$ is represented by the curve $s \mapsto c(s)d(s)$. Furthermore, the zero section $G \rightarrow TG$ is a homomorphism, and we write $g[c] = [s \mapsto gc(s)]$ for the left action of G on TG . For classical Lie groups G , there are two closely related ways to trivialize T^2G . The first one is to trivialize $TG \cong G \times \mathfrak{g}$, and then apply T to get $T(G \times \mathfrak{g}) \cong TG \times \mathfrak{g} \times T_0\mathfrak{g}$, then trivialize the first factor. The second way is to regard TG as a Lie group, trivialize $TTG \cong TG \times T_0(TG)$ and then trivialize TG . See [5] for the precise relation of these trivializations.

We will use the first approach. The projection $\pi : TG \rightarrow G$ which maps $[c]$ to $c(0)$ is a group homomorphism. For any $v \in TG$ one has $\pi(v)^{-1}v \in \mathfrak{g}$. Define

$$\Phi : TG \rightarrow G \times \mathfrak{g}$$

by $\Phi([c]) := (c(0), c(0)^{-1}[c])$.

Lemma 3.1. *The map Φ is an isomorphism of Frölicher groups, when $G \times \mathfrak{g}$ carries the semidirect product group structure with multiplication*

$$(g, v)(h, w) := (gh, \text{Ad}(h^{-1})v + w).$$

Proof. Clearly, Φ is smooth and bijective. The inverse of Φ is given by $(g, v) \mapsto gv$, which is also smooth. It remains to show that Φ is a homomorphism. By definition we have

$$\Phi([c][d]) = (c(0)d(0), d(0)^{-1}c(0)^{-1}[c][d]).$$

Let us write $g = c(0)$ and $h = d(0)$. The second component of above expression can be written as

$$h^{-1}g^{-1}[c]hh^{-1}[d].$$

By Lemma 2.3, this is the sum $h^{-1}g^{-1}[c]h + h^{-1}[d] = \text{Ad}(h^{-1})g^{-1}[c] + h^{-1}[d]$, and Φ is a homomorphism. \square

The map $T\Phi$ is an isomorphism between T^2G and $T(G \times \mathfrak{g})$. Further, $T(G \times \mathfrak{g})$ is isomorphic to $TG \times T\mathfrak{g}$, and since G and \mathfrak{g} are Frölicher groups, we can further trivialize to get an isomorphism, which we also denote as Φ , between T^2G and $G \times \mathfrak{g} \times \mathfrak{g} \times T_0\mathfrak{g}$. If $\xi \in T^2G$, let

$$\Phi(\xi) =: (\pi_1(\xi), \pi_2(\xi), \pi_3(\xi), \pi_4(\xi)).$$

We now describe the components $\pi_i(\xi)$.

Lemma 3.2. *Suppose $\gamma : \mathbb{R}^2 \rightarrow G$ represents $\xi \in T^2G$. Let $g := \gamma(0, 0)$. Then we have*

$$\begin{aligned} \pi_1(\xi) &= g \\ \pi_2(\xi) &= [s \mapsto g^{-1}\gamma(s, 0)] \\ \pi_3(\xi) &= [t \mapsto g^{-1}\gamma(0, t)] \\ \pi_4(\xi) &= [s \mapsto [t \mapsto \gamma(0, t)^{-1}g\gamma(s, 0)^{-1}\gamma(s, t)]] \in T_0\mathfrak{g}. \end{aligned}$$

Proof. We first apply $T\Phi$:

$$\begin{aligned} T\Phi(\xi) &= [s \mapsto \Phi([t \mapsto \gamma(s, t)])] \\ &= [s \mapsto (\gamma(s, 0), \gamma(s, 0)^{-1}[t \mapsto \gamma(s, t)])] \\ &= [s \mapsto (\gamma(s, 0), [t \mapsto \gamma(s, 0)^{-1}\gamma(s, t)])]. \end{aligned}$$

We use the canonical identification $T(G \times \mathfrak{g}) \cong TG \times T\mathfrak{g}$. Then we use the trivializations of TG and $T\mathfrak{g}$, respectively. Clearly, the part in $TG = G \times \mathfrak{g}$ is given by

$$(\gamma(0, 0), [s \mapsto \gamma(0, 0)^{-1}\gamma(s, 0)]),$$

which proves the formula for π_1 and π_2 . Similarly, the formula for π_3 and π_4 is obtained by applying the trivialization $T\mathfrak{g} \rightarrow \mathfrak{g} \times T_0\mathfrak{g}$ to the element of $T\mathfrak{g}$ represented by the function $\gamma(s, 0)^{-1}\gamma(s, t)$. \square

Corollary 3.3. *Let $[c], [d] \in \mathfrak{g}$ and let $\gamma(s, t) := c(s)d(t)c(s)^{-1}d(t)^{-1}$. Then the trivialization of T^2G maps the element $\xi \in T^2G$ represented by γ to*

$$(e, 0, 0, [s \mapsto [t \mapsto \gamma(s, t)]]).$$

Proof. This follows since $\gamma(s, 0) = \gamma(0, t) = e$ for all s and t . \square

Remark 3.4. If $v = [c] \in \mathfrak{g}$, one easily checks that $\gamma_1(s, t) := c(s)$ and $\gamma_2(s, t) := c(t)$ satisfy the conditions of Lemma 2.9. Let $\iota_1(v)$ and $\iota_2(v)$ denote the corresponding elements of T^2G . This yields canonical maps $\iota_i : \mathfrak{g} \rightarrow T^2G$, which in above trivialization of T^2G correspond to the inclusions of \mathfrak{g} as the second and third factors of $G \times \mathfrak{g} \times \mathfrak{g} \times T_0\mathfrak{g}$, respectively. Let $\kappa : G \times G \rightarrow G$ denote the commutator map $\kappa(g, h) := ghg^{-1}h^{-1}$. Then it follows from Remark 2.8 that $T^2\kappa(\iota_1(v), \iota_2(v))$ is represented by

$$\gamma(s, t) = \kappa(\gamma_1(s, t), \gamma_2(s, t)) = c(s)d(t)c(s)^{-1}d(t)^{-1}.$$

Lie bracket. Under the assumption that a natural map $\mathfrak{g} \rightarrow T_0\mathfrak{g}$ is bijective we can define a bilinear bracket operation $[\cdot, \cdot]$ on \mathfrak{g} . The goal of this subsection is to prove that $[\cdot, \cdot]$ is a Lie bracket.

Definition 3.5. (a) Suppose that V is a vector space with a Frölicher structure such that scalar multiplication is smooth. We define a map $\Xi : V \rightarrow T_0V$ by sending v to $[s \mapsto sv]$, the line through v . If $V := T_x X \subset TX$ for some Frölicher space X , and V carries the subspace structure, then it is easily checked that Ξ is injective.

(b) Let G be a Frölicher group. The tangent space $\mathfrak{g} = T_e G$ inherits a Frölicher structure from TG . Suppose that the map $\Xi : \mathfrak{g} \rightarrow T_0\mathfrak{g}$ is bijective. Suppose that $v = [c], w = [d] \in \mathfrak{g}$, and let $\xi \in T_0\mathfrak{g}$ be represented by $\gamma(s, t) = c(s)d(t)c(s)^{-1}d(t)^{-1}$. Then we define

$$[v, w] := \Xi^{-1}(\xi).$$

Here is an auxiliary lemma which will be needed later.

Lemma 3.6. *Let X and Y be Frölicher spaces, such that the maps Ξ corresponding to $T_x X$ and $T_y Y$ are bijective. Let $\varphi : X \rightarrow Y$ be smooth. Then*

$$T\varphi(\Xi^{-1}(\xi)) = \Xi^{-1}(T^2\varphi(\xi)).$$

Proof. Suppose that $\Xi^{-1}(\xi) \in T_x X$ is represented by $s \mapsto c(s)$. Then $T\varphi(\Xi^{-1}(\xi))$ is represented by $\varphi \circ c$. Now we apply Ξ and get an element of $T_0T_x X$ represented by $\gamma(s, t) = \varphi(c(st))$. By Remark 2.8 this map also represents $T^2\varphi(\xi)$, which proves our claim. \square

Our goal is to prove that $[\cdot, \cdot]$ is a Lie bracket on \mathfrak{g} . To this end, we view elements of \mathfrak{g} as derivations of the algebra F of functions on G . It will then suffice to show that $[v, w]$ corresponds to the commutator of the derivations corresponding to v and w .

Definition 3.7. We let $\lambda_g : G \rightarrow G$ denote left multiplication by $g \in G$. If $v = [c] \in \mathfrak{g}$, we define $D_v : F \rightarrow F$ by

$$D_v f(g) := b(v, [f \circ \lambda_g]) = (f \circ \lambda_g \circ c)'(0).$$

Lemma 3.8. For each $v \in \mathfrak{g}$, the map D_v is a derivation of the \mathbb{R} -algebra F . The map $v \mapsto D_v$ is a linear injection of \mathfrak{g} into the vector space of derivations of F .

Proof. Suppose that $v = [c] \in \mathfrak{g}$ and $f, g \in F$. The vector space F is an algebra under pointwise multiplication of functions. Therefore, $D_v(fg)(x)$ is the derivative of the product $(f \circ \lambda_x \circ c)(g \circ \lambda_x \circ c)$ of functions in $C^\infty(\mathbb{R}, \mathbb{R})$, and the product rule for derivatives shows that $D_v(fg)(x) = f(x)D_v(g)(x) + g(x)D_v(f)(x)$.

Now suppose that $v = [c] \neq w = [d]$ in \mathfrak{g} . By definition, this means that there is some $f \in F$ with $(f \circ c)'(0) \neq (f \circ d)'(0)$; hence $D_v f(e) \neq D_w f(e)$, proving injectivity of $v \mapsto D_v$.

Linearity in v follows from $D_v f(x) = b(v, [f \circ \lambda_x])$. \square

Let $[D_v, D_w]$ be the usual commutator of derivations. If

$$D_{[v, w]} = [D_v, D_w] \tag{3.1}$$

for all v and w , then it follows immediately that $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra. The rest of this section is devoted to proving Eq. (3.1).

Let us first compute the left hand side of Eq. (3.1).

Lemma 3.9. Suppose that $\Xi : \mathfrak{g} \rightarrow T_0\mathfrak{g}$ is a bijection. Let $\xi \in T_0\mathfrak{g}$ be represented by $\gamma : \mathbb{R}^2 \rightarrow G$. If $v = \Xi^{-1}(\xi)$ and $f \in F$, then

$$D_v(f)(e) = \frac{\partial^2(f \circ \gamma)}{\partial s \partial t}(0, 0).$$

Proof. Suppose that $v = [c]$. We have seen that then, $\xi = \Xi(v)$ can be represented by $\rho(s, t) = c(st)$. Let $\bar{\gamma}(s) = [t \mapsto \gamma(s, t)]$ and $\bar{\rho}(s) = [t \mapsto c(st)]$. By definition of $T_0\mathfrak{g}$, for every smooth function $\varphi : \mathfrak{g} \rightarrow \mathbb{R}$ we have

$$(\varphi \circ \bar{\rho})'(0) = (\varphi \circ \bar{\gamma})'(0). \tag{3.2}$$

We can apply this if φ is the second component of $Tf : TG \rightarrow \mathbb{R}^2$ for $f \in F$, to get

$$\pi_2(Tf(\bar{\rho}(s))) = (f \circ \rho(s, \cdot))'(0) = \frac{(\partial f \circ \rho)}{\partial t}(s, 0).$$

Hence, with $\varphi = \pi_2 \circ Tf$, the left hand side of Eq. (3.2) becomes

$$\frac{\partial^2(f \circ \rho)}{\partial s \partial t}(0, 0).$$

We have $\rho(s, t) = c(st)$, so the chain rule yields $\frac{\partial}{\partial t}\rho(s, t) = s(f \circ c)'(st)$. Now we use the product rule to get

$$\frac{\partial^2(f \circ \rho)}{\partial s \partial t}(s, t) = (f \circ c)'(st) + s^2(f \circ c)''(st).$$

We let $(s, t) = (0, 0)$ and equate with the right hand side of Eq. (3.2) to get

$$(f \circ c)'(0) = \frac{\partial^2(f \circ \gamma)}{\partial s \partial t}(0, 0),$$

which is what we claimed. \square

Now we turn to the right hand side of Eq. (3.1).

Lemma 3.10. *If $v = [c]$, $w = [d] \in \mathfrak{g}$, then*

$$D_v(D_w f)(e) = \frac{\partial^2}{\partial s \partial t} \Big|_{(s,t)=(0,0)} f(c(s)d(t)).$$

Proof. By definition, $D_w f(g) = (f \circ \lambda_g \circ d)'(0)$. Note that λ_e is the identity map on G , so $D_v(D_w f)(e)$ is the derivative of

$$s \mapsto D_w f(c(s)) = (f \circ \lambda_{c(s)} \circ d)'(0)$$

at 0. Now $f(\lambda_{c(s)}(d(t))) = f(c(s)d(t))$; hence we obtain

$$D_v(D_w f)(e) = \frac{\partial}{\partial t} \Big|_{t=0} \left(\frac{\partial}{\partial s} \Big|_{s=0} f(c(s)d(t)) \right),$$

as was claimed. \square

Our last step now is to show that left hand side and right hand side agree.

Lemma 3.11. *If $v = [c]$, $w = [d] \in \mathfrak{g}$ and $\gamma(s, t) = c(s)d(t)c(s)^{-1}d(t)^{-1}$, then*

$$\frac{\partial^2}{\partial s \partial t} \Big|_{(s,t)=(0,0)} f \circ \gamma = \frac{\partial^2}{\partial s \partial t} \Big|_{(s,t)=(0,0)} (f(c(s)d(t)) - f(d(s)c(t))).$$

Proof. First we fix the variable s and use the formulas (2.1) for inversion and addition in $T_{c(s)}G$ to obtain

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} (f(c(s)d(t)) - f(d(t)c(s))) &= \frac{\partial}{\partial t} \Big|_{t=0} (f(c(s)d(t)) + f(d(t)^{-1}c(s))) \\ &= \frac{\partial}{\partial t} \Big|_{t=0} (f(c(s)d(t))c(s)^{-1}d(t)^{-1}c(s)) \\ &= \frac{\partial}{\partial t} \Big|_{t=0} (f(\gamma(s, t)c(s))). \end{aligned}$$

Now fix t and consider the curve $s \mapsto \gamma(s, t)c(s)$ through e . We use the formula for addition in $T_e G$ to get

$$\frac{\partial}{\partial s} \Big|_{s=0} (f(\gamma(s, t)c(s))) = \frac{\partial}{\partial s} \Big|_{s=0} (f(\gamma(s, t)) + f(c(s))).$$

Finally we get the desired result by differentiating the last expression with respect to t . \square

We combine our results to get the following:

Theorem 3.12. *Let G be a Frölicher group, $\mathfrak{g} = T_e G$ and $\Xi : \mathfrak{g} \rightarrow T_0 \mathfrak{g}$ the map sending v to $[s \mapsto sv]$. If Ξ is bijective, then $[\cdot, \cdot]$ as in Definition 3.5 is a Lie bracket. Let H be another Frölicher group for which $\mathfrak{h} = T_e H$ has the property that $\Xi : \mathfrak{h} \rightarrow T_0 \mathfrak{h}$ is bijective. If $\alpha : G \rightarrow H$ is a smooth homomorphism, then $T\alpha$ is a morphism of Lie algebras.*

Proof. We have shown that the map $v \mapsto D_v$ in Definition 3.7 is linear, injective and satisfies $D_{[v,w]} = [D_v, D_w]$. This immediately implies that $[\cdot, \cdot]$ is a Lie bracket on \mathfrak{g} . Let $\alpha : G \rightarrow H$ be a morphism of Frölicher groups, where H is another group for which $\Xi : \mathfrak{h} \rightarrow T_0 \mathfrak{h}$ is bijective. We claim that $T\alpha$ respects the Lie bracket. Suppose that $v = [c], w = [d] \in \mathfrak{g}$ and let $\xi \in T^2 G$ be represented by $\gamma(s, t) = c(s)d(t)c(s)^{-1}d(t)^{-1}$. We have seen in Lemma 3.6 that $T\alpha([v, w]) = T\alpha(\Xi^{-1}(\xi)) = \Xi^{-1}(T^2\alpha(\xi))$, and by Remark 2.8, $T^2\alpha(\xi)$ is represented by $\alpha \circ \gamma$. Now α is a homomorphism; therefore $\alpha(\gamma(s, t))$ is the commutator of $\alpha(c(s))$ and $\alpha(d(t))$. This proves

$$T\alpha([v, w]) = [T\alpha(v), T\alpha(w)]. \quad \square$$

4. An example

As an example for the theory developed above, we consider the product group $G = (\mathbb{R}^J, +)$ with its product Frölicher structure as described in Example 1.23.

The vector spaces \mathbb{R}^J with their product topology play an important role in the theory of pro-Lie groups; see Hofmann and Morris [14, Appendix 2] for a summary of results. Their topological properties depend on the cardinality of J . For instance, \mathbb{R}^J is polish (that is, separable and completely metrizable) if and only if J is at most countable. It is separable if and only if the cardinality of J is less than or equal to that of \mathbb{R} .

Here we will show that under a rather weak assumption on the cardinality of J , the Frölicher spaces \mathbb{R}^J and $T_0 \mathbb{R}^J$ are isomorphic. The crucial idea is that functions $f : \mathbb{R}^J \rightarrow \mathbb{R}$ of the Frölicher structure factor through \mathbb{R}^{J_0} with countable J_0 . Since \mathbb{R}^{J_0} is a Fréchet space, this allows us to regard them as functions of class C^∞ on \mathbb{R}^{J_0} . The assumption made on J is that its cardinality is strictly smaller than that of any *weakly inaccessible cardinal*. We will use the Appendix to indicate which cardinals satisfy this assumption. Let us just mention that the existence of weakly inaccessible cardinals cannot be proven within the standard axiom system ZFC. Therefore, the following lemma is true for most J that naturally occur.

Proposition 4.1. *Let J be a set of cardinality smaller than any weakly inaccessible cardinal. Equip \mathbb{R}^J with the product Frölicher structure (C, F) . Then $f \in F$ if and only if there is a countable $J_0 \subset J$ such that f factors through a map $f_0 : \mathbb{R}^{J_0} \rightarrow \mathbb{R}$, and the map f_0 is of class C^∞ for the product topology on \mathbb{R}^{J_0} .*

Proof. If f factors through a C^∞ -map $f_0 : \mathbb{R}^{J_0} \rightarrow \mathbb{R}$, it is clearly smooth along smooth curves, and hence in F . Now conversely, if $f \in F$, then f is continuous in the c^∞ -topology of \mathbb{R}^J , and it was shown in [15, Example I.4.27] that f factors as $f_0 : \mathbb{R}^{J_0} \rightarrow \mathbb{R}$ with countable J_0 (see Appendix for an outline). Now f_0 is smooth along smooth curves, and hence by Corollary 1.12, it is of class C^∞ . \square

Theorem 4.2. *If the cardinality of J is smaller than any weakly inaccessible cardinal, then*

$$T_0 \mathbb{R}^J \cong \mathbb{R}^J.$$

Proof. Let C_0 denote the set of curves through 0. Then we define $\varphi : C_0 \rightarrow \mathbb{R}^J$ by $\varphi(c)_j = c'_j(0)$. This map is clearly onto. We need to see that it factors through a bijection $T_0\mathbb{R}^J \rightarrow \mathbb{R}^J$.

Claim: $\ker \varphi = \{c \in C_0 \mid \forall f \in F : (f \circ c)'(0) = 0\}$.

The inclusion \supseteq is clear, using $f = \pi_j$. To prove the reverse inclusion, suppose that $f \in F$ and $c \in \ker \varphi$. Then f factors as $f = f_0 \circ \pi$, where $\pi : \mathbb{R}^J \rightarrow \mathbb{R}^{J_0}$ with countable $J_0 \subset J$. Hence $(f \circ c)'(0) = (f_0 \circ \pi \circ c)'(0) = d_0 f((\pi \circ c)'(0))$ by the chain rule for C^∞ -maps. Now $(\pi \circ c)'(0) = 0$, since otherwise if the i th component is non-zero, we use that $\pi_i \circ \pi \circ c = \pi_i \circ c$ to conclude that $(\pi_i \circ c)'(0) \neq 0$, contradicting the choice of c . Hence $(f \circ c)'(0) = 0$.

Claim: The induced bijection $\bar{\varphi} : T_0\mathbb{R}^J \rightarrow \mathbb{R}^J$ is smooth and has a smooth inverse.

To see that $\bar{\varphi}$ is smooth, compose it with π_j to get $\pi_j \circ \bar{\varphi}([c]) = (\pi_j \circ c)'(0)$. But this is the second component of $T\pi_j([c]) \in T\mathbb{R} = \mathbb{R}^2$. Now $T\pi_j$ is smooth on $T_0\mathbb{R}^J$; hence $\pi_j \circ \bar{\varphi}$ is smooth. Now consider $\bar{\varphi}^{-1}$. Given $x = (x_j)_{j \in J} \in \mathbb{R}^J$, a representative of $\bar{\varphi}^{-1}(x)$ is given by $[d]$ where $d_j(t) = tx_j$. Suppose that $c \in C$ and $f \in F$. Then

$$T_0 f \circ \bar{\varphi}^{-1} \circ c : s \mapsto T_0 f([t \mapsto (tc_j(s))_{j \in J}]).$$

Again we use the fact that f factors through a countable product \mathbb{R}^{J_0} . We identify $[t \mapsto (tc_j(s))_{j \in J_0}] \in T_0\mathbb{R}^{J_0}$ with $(c_j(s))_{j \in J_0} \in \mathbb{R}^{J_0}$ and obtain

$$(T_0 f \circ \bar{\varphi}^{-1} \circ c)(s) = d_0 f((c_j(s))_{j \in J_0}),$$

which is smooth in s as a composition of C^∞ -functions. \square

Concluding remarks. It would be desirable to verify $T_0\mathfrak{g} \cong \mathfrak{g}$ for at least the following classes of Frölicher groups:

- $\mathcal{G} = C^\infty(M, G)$, in particular for non-compact manifolds M . The difficulty here, as in the case of \mathbb{R}^J , is to determine the class of smooth functions on \mathcal{G} . By cartesian closedness, c is a smooth \mathcal{G} -valued curve if and only if $\tilde{c} : \mathbb{R} \times M \rightarrow G$ is smooth. If c is a curve through e , we get a map $m \mapsto (\tilde{c}(\cdot, m))'(0) \in \mathfrak{g}$. If one knew the class of functions, one might be able to show that this assignment yields a map $T_e\mathcal{G} \rightarrow C^\infty(M, \mathfrak{g})$.
- $G = \text{Diff}(M)$. Here, the difficulty is again in finding the class of functions $f : G \rightarrow \mathbb{R}$.
- Pro-Lie groups, and as a test case, the p -adic solenoid \mathbb{T}_p . This is the projective limit of the diagram

$$\mathbb{S}^1 \xleftarrow{\varphi} \mathbb{S}^1 \xleftarrow{\varphi} \mathbb{S}^1 \dots$$

with $\varphi(z) = z^p$. The projective limit of this diagram in the category of Hausdorff topological groups is a *topological group with Lie algebra* (see [14]), and there is a continuous exponential function $\exp : \mathbb{R} \rightarrow \mathbb{T}_p$. This map is easily seen to be smooth, and this fact can be used to conclude that $T_e\mathbb{T}_p \cong \mathbb{R}$ as vector spaces. However, we have not been able to determine the Frölicher structure on this vector space, so we were not able to show $T_0\mathbb{R} \cong \mathbb{R}$. A better knowledge of the set F of functions on \mathbb{T}_p would be necessary.

The examples of \mathbb{R}^J and \mathbb{T}_p seem to indicate that progress is not possible without using a topology on the Frölicher group under consideration. It is not clear to the author whether this has to be done on a case-by-case basis, or whether one should use a topology associated with the Frölicher structure. For example, Michor suggested restricting attention to Frölicher spaces (X, C, F) for which the initial topology with respect to F agrees with the final topology with respect to C .

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Appendix

In this appendix we provide details for the proof of [Proposition 4.1](#).

If J is any index set, let \mathbb{R}^J and 2^J carry the product topology, and let \mathbb{R}_0^J and 2_0^J denote the subsets of elements with countable support. Consider the following two properties, which we think of as properties of the cardinality $\kappa = |J|$.

- (A) If $f : \mathbb{R}_0^J \rightarrow \mathbb{R}$ is sequentially continuous, then f factors through a subspace \mathbb{R}^{J_0} with countable $J_0 \subset J$.
- (B) If $f : 2^J \rightarrow \mathbb{R}$ is sequentially continuous and vanishes on 2_0^J , then $f = 0$.

Recall that cardinals can be well-ordered and indexed by ordinals: \aleph_0 is the first infinite cardinal and for any ordinal $\alpha > 0$; the symbol \aleph_α denotes the α th uncountable cardinal.

Mazur proves the following theorem in [18].

Theorem A.1. • \aleph_0 satisfies (A) and (B).

- If \aleph_α satisfies (A) or (B), so does $\aleph_{\alpha+1}$.
- If the κ_s , $s \in S$, and the cardinality $|S|$ satisfy (A) or (B), so does the union $\kappa = \bigcup_{s \in S} \kappa_s$.

This says that (A) and (B) hold for all cardinalities smaller than the first weakly inaccessible cardinal.

We now sketch the proof of Kriegel and Michor's argument. They use it to show that the c^∞ -topology of \mathbb{R}^J is not completely regular if $|J| > |\mathbb{R}|$. Note that being Hausdorff and completely regular is equivalent to the existence of a uniform structure inducing the topology; hence \mathbb{R}^J does not have a uniform structure inducing the c^∞ -topology. In particular, \mathbb{R}^J with the c^∞ -topology is not a topological group.

Lemma A.2. Let the cardinality $|J|$ satisfy (A) and (B) above. Suppose that $f : \mathbb{R}^J \rightarrow \mathbb{R}$ is c^∞ -continuous. Then f factors through a countable product \mathbb{R}^{J_0} .

Proof. If $A \subset J$ and $x \in \mathbb{R}^J$, let x_A be defined by

$$(x_A)_j = \begin{cases} x_j & \text{if } j \in A, \\ 0 & \text{else.} \end{cases}$$

Now if $f : \mathbb{R}^J \rightarrow \mathbb{R}$ is continuous, its restriction to \mathbb{R}_0^J is continuous. On \mathbb{R}_0^J , the c^∞ -topology agrees with the restriction of the product topology on \mathbb{R}^J (see [15, Example I.4.12]). Thus by property (A), the restriction $f|_{\mathbb{R}_0^J}$ depends only on countably many variables $J_0 \subset J$, that is, $f(x) = f(x_{J_0})$ for all $x \in \mathbb{R}_0^J$. We claim that $f(x) = f(x_{J_0})$ for all $x \in \mathbb{R}^J$. To this end, fix $x \in \mathbb{R}^J$. We view subsets $A \subset J$ as elements of the topological space 2^J , and define a function

$\varphi_x(A) = f(x_A) - f(x_{A \cap J_0})$ from 2^J to \mathbb{R} . By the previous argument, $\varphi_x(A) = 0$ for countable A . We are done if we show that φ_x is sequentially continuous, since then by property (B) it follows that $\varphi_x(J) = 0$, hence $f(x) = f(x_{J_0})$.

In order to prove sequential continuity, suppose that $A_n \rightarrow A$ in 2^J . Then one can show that $\{n(x_{A_n} - x_A) | n \in \mathbb{N}\}$ is a bounded subset of the topological vector space \mathbb{R}^J . This implies that x_{A_n} is Mackey-convergent to x_A , and hence also convergent in the c^∞ -topology (we refer the reader to [15, Chapter I, Section 4.9] for details on this point). The map f is c^∞ -continuous, and therefore $\varphi_x(A_n) \rightarrow \varphi_x(A)$ as $n \rightarrow \infty$. \square

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